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Fall Term
Week 3

Limits & Continuity

This exercise sheet consists of two parts: at first additional exercises are given the solutions of which are provided with the lecture slides and can serve you as further blueprints when solving similar tasks. Then, the actual homework assignments are stated. Please, hand-in your results of the homework assignments through MSTeams (see there for further details).

Additional Exercises (see the lecture slides for solutions):

Exercise 3.1: Find $\lim_{x\to 4} f(x)$ (if it exists) where

$$f(x) \;=\; \begin{cases} \sqrt{x-4} & \text{if } x > 4 \\ 8-2x & \text{if } x < 4 \end{cases}$$

Hereby follow these steps to the solution: First, determine the right-hand side limit $\lim_{x\to 4^+} f(x)$. Then, determine the left-hand side limit $\lim_{x\to 4^-} f(x)$. Finally, the limit exists if the two one-sided limits have the same value.

Exercise 3.2: Determine the following limits (if it exists, or the way it does not exist)

$$\lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x} .$$

Hint: Apply the Third Binomial Formula after multiplying both numerator and denominator by $\sqrt{1+x}+\sqrt{1-x}$ (recall, this procedure is called rationalizing).

Exercise 3.3: Let

$$f(x) \;=\; \frac{x^2-5x+6}{x^2-2x} .$$

Does the graph of f have any vertical asymptotes?

Exercise 3.4: Sketch the graph of

$$f(x) \;=\; (x-2)^4 \cdot (x+1)^3 \cdot (x-1)$$

by first discussing the continuity of f, finding its roots (x-intercepts), y-intercept, and limits at infinity as $x \to \infty$ and $x \to -\infty$.

Exercise 3.5: Draw the graph of a function $y = f(x)$ such that the following properties are satisfied:

a) $\lim\limits_{x\to -2} f(x) \;=\; 1$ c) $\lim\limits_{x\to 2^+} f(x) \;=\; 2$ e) $f(2) \;=\; 1$

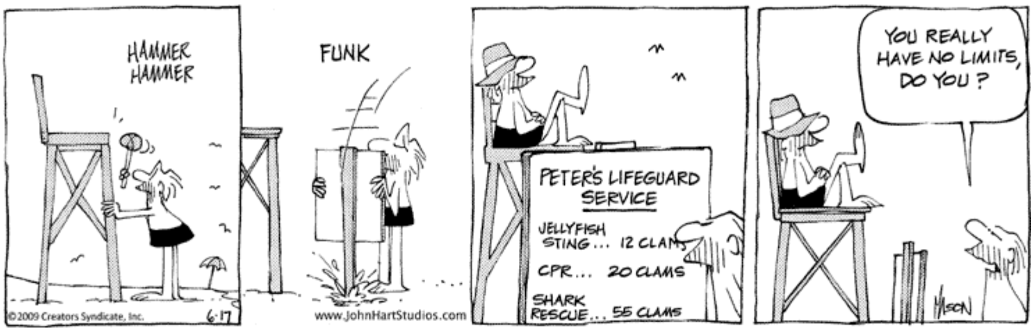
b) $\lim\limits_{x\to 0} f(x) \;=\; \infty$ d) $\lim\limits_{x\to 2^-} f(x) \;=\; 0$ f) $\lim\limits_{x\to \infty} f(x) \;=\; 1$

Exercise 3.6: Show that the function $f(x) = 1-\sqrt{1-x^2}$ is continuous on the closed interval $[-1, 1]$.

Exercise 3.7: Show that there is a root of the equation

$$f(x) \;=\; 0 \qquad \text{with} \qquad f(x) \;=\; 4x^3-6x^2+3x-2$$

between 1 and 2.



<u>**Homework Assignment:**</u>

Problem 3.1: Limits.

a) Find the indicated limit if it exists

$$\lim_{x\to -1}(x^3-2x^2+x-3)\,, \qquad \lim_{x\to \frac{1}{3}}\frac{x+1}{x+2}\,, \qquad \lim_{x\to 2}\frac{x^2+x-6}{x-2}\,, \qquad \text{and} \qquad \lim_{x\to 9}\frac{\sqrt{x}-3}{x-9}\,.$$

b) Find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$. If the limiting value is infinite, indicate whether it is $+\infty$ or $-\infty$.

$$f(x) \;=\; x^3-4x^2-4\,, \qquad f(x) \;=\; \frac{x^2-2x+3}{2x^2+4x+1}\,, \qquad \text{and} \qquad f(x) \;=\; \frac{1-2x^3}{x+1}\,.$$

c) Find the limits

$$\lim_{x\to c}(2f(x)-3g(x)) \qquad \text{and} \qquad \lim_{x\to c}\frac{2f(x)-g(x)}{5g(x)+2f(x)}$$

or show that it does not exist using the following facts about limits involving the functions $f(x)$ and $g(x)$:

$$\lim_{x\to c} f(x) \;=\; 5 \qquad \lim_{x\to c} g(x) \;=\; -2$$

Problem 3.2: One-sided limits.

a) Find the indicated one-sided limit. If the limiting value is infinite, indicate whether it is $+\infty$ or $-\infty$.

$$\lim_{x\to 4^+}(3x^2-9)\,, \qquad \lim_{x\to 1^-}\frac{x-\sqrt{x}}{x-1}\,, \qquad \text{and} \qquad \lim_{x\to 5^+}\frac{\sqrt{2x-1}-3}{x-5}\,.$$

b) Find the one-sided limits $\lim_{x\to 3^-} f(x)$ and $\lim_{x\to 3^+} f(x)$, where

$$f(x) \;=\; \begin{cases} 2x^2-x & \text{if } x<3 \\ 3-x & \text{if } x\geq 3 \end{cases}$$

If the limiting value is infinite, indicate whether it is $+\infty$ or $-\infty$.

c) Find the one-sided limits $\lim_{x\to 1^-} f(x)$ and $\lim_{x\to 1^+} f(x)$, where

$$f(x) \;=\; \begin{cases} \frac{1}{x-1} & \text{if } x<3 \\ x^2+2x & \text{if } x\geq 3 \end{cases}$$

If the limiting value is infinite, indicate whether it is $+\infty$ or $-\infty$.

Problem 3.3: Continuity.

a) Decide if the given function is continuous at the specified value of x.

(i) $f(x) = x^3 - 2x^2 + x - 5$ at $x = 0$.

(ii) $f(x) = \frac{2x+1}{3x-6}$ at $x = 2$.

(iii) $f(x) = \frac{\sqrt{x}-2}{x-4}$ at $x = 4$.

b) Decide if the given function is continuous at the specified value of x.

$$f(x) = \begin{cases} x+1 & \text{if } x \le 2 \\ 2 & \text{if } x > 2 \end{cases}, \qquad \text{at } x = 2\,.$$

c) List all the values of x for which the given functions is not continuous.

$$f(x) = x^5 - x^3\,, \qquad f(x) = \frac{x^2 - 2x + 1}{x^2 - x - 2}\,,$$

as well as

$$f(x) = \begin{cases} 2 - 3x & \text{if } x \le -1 \\ x^2 - x + 3 & \text{if } x > 1 \end{cases}.$$

Problem 3.4: Applications in Business, Economics, and Management.

a) Tomas, the organizer of a sports event, estimates that if the event is announced x days in advance, the revenue obtained will be $R(x)$ thousand GEL, where

$$R(x) = 400 + 120x - x^2\,.$$

The cost of advertising the event for x days is $C(x)$ thousand GEL, where

$$C(x) = 2x^2 + 300\,.$$

(i) Find the profit function $P(x) = R(x) - C(x)$, and sketch its graph.

(ii) How many days in advance should Tomas announce the event to maximize profit? What is the maximum profit?

(iii) What is the ratio of revenue to cost

$$Q(x) = \frac{R(x)}{C(x)}$$

at the optimal announcement time found in part **(ii)**? What happens to this ratio as $x \to 0$? Interpret these results.

b) Alicia, the manager of a plant, determines that when x% of the plant's capacity is being used, the total cost of operation is C thousand GEL, where

$$C(x) = \frac{8x^2 - 636x - 320}{x^2 - 68x - 960}\,.$$

The company has a policy of rotating maintenance in an attempt to ensure that approximately 80% of capacity is always in use.

(i) What cost should Alicia expect when the plant is operating at this ideal capacity?

(ii) Find $C(0)$ and $C(100)$.

(iii) Explain why Alicia cannot use the result of part **(ii)** along with the intermediate value property to conclude that the cost of operation is exactly 700000 GEL when a certain percentage of plant capacity is being used.

c) In certain situations, it is necessary to weigh the benefit of pursuing a certain goal against the cost of achieving that goal. For instance, suppose that to remove x% of the pollution from an oil spill, it costs C thousands of GEL, where

(i) How much does it cost to remove 25% of the pollution? 50%?

(ii) Sketch the graph of the cost function.

(iii) What happens as $x \to 100^-$? Is it possible to remove all the pollution?