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Fall Term
Week 4

Derivatives & Techniques of Differentiation

This exercise sheet consists of two parts: at first additional exercises are given the solutions of which are provided with the lecture slides and can serve you as further blueprints when solving similar tasks. Then, the actual homework assignments are stated. Please, hand-in your results of the homework assignments through MSTeams at the date and time specified in MSTeams.

Additional Exercises (see the lecture slides for solutions):

Exercise 4.1: Tangent lines.

- **a)** Find the slope of the tangent line to the curve $y = x - x^3$ at the point $(1, 0)$.
- **b)** Find an equation of the tangent line in part **a)**.
- **c)** Graph the curve and the tangent line in successively smaller viewing rectangles centered at $(1, 0)$ until the curve and the line appear to coincide.

Exercise 4.2: The graph of a function f is shown in figure 1 (a).

- **a)** Find the average rate of change of f on the interval $[20, 60]$.
- **b)** Identify an interval on which the average rate of change of f is 0.
- **c)** Which interval gives a larger average rate of change, $[40, 60]$ or $[40, 70]$?
- **d)** Compute
$$\frac{f(40) - f(10)}{40 - 10};$$
what does this value represent geometrically?

Exercise 4.3: Let the graph of f be given as in figure 1 (b). State, with reasons, the numbers at which f is not differentiable

Exercise 4.4: Find both the derivative with respect to x, i.e. $\frac{\mathrm{d} f(x,t)}{\mathrm{d} x}$, and t, i.e. $\frac{\mathrm{d} f(x,t)}{\mathrm{d} t}$, for
$$f(x, t) = \frac{t}{x^2} + \frac{x}{t}.$$

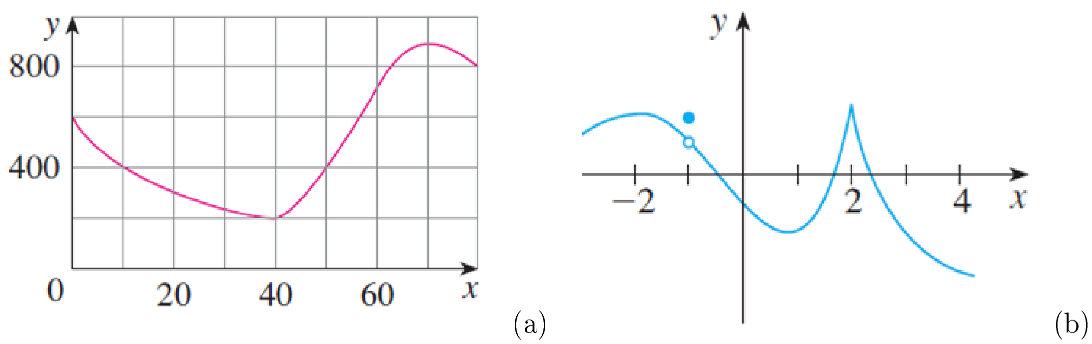

Figure 1: Sketches to be used in exercises 4.2 and 4.3.

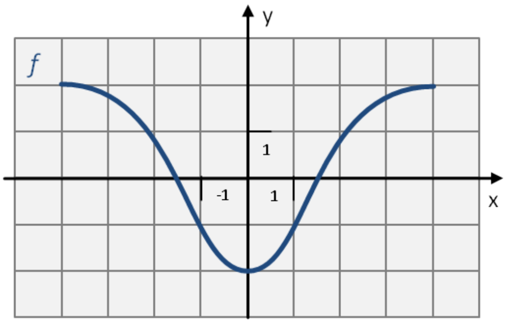

Figure 2: Sketches to be used in exercise 4.6.

Exercise 4.5: Sketch the graph of a function g for which the following properties hold:

$$g(0) \;=\; g(2) \;=\; g(4) \;=\; 0\,,$$

and

$$g'(1) \;=\; g'(3) \;=\; 0\,, \qquad g'(0) \;=\; g'(4) \;=\; 1\,,$$

and

$$g'(2) \;=\; -1\,,$$

as well as

$$\lim_{x\to\infty} g(x) \;=\; \infty\,, \qquad \text{and} \qquad \lim_{x\to-\infty} g(x) \;=\; -\infty\,.$$

Exercise 4.6: Trace or copy the graph of the function f given in figure 2. (Assume that the axes have equal scales.) Then estimate the derivative at certain points of the graph of f to sketch the graph of f' below it.

Homework Assignment:

Problem 4.1: Basic concepts of differentiation - I.

a) Compute the derivative of the given function and find the slope as well as the equation of the line that is tangent to its graph for the specified value of the independent variable x:

$$\textbf{(i)}\quad f(x) \;=\; 4\,, \quad \text{at } x=0\,; \qquad \textbf{(ii)}\quad f(x) \;=\; 2-7x\,, \quad \text{at } x=-1\,;$$

as well as

$$\textbf{(iii)}\quad f(x) \;=\; x^{-2}\,, \quad \text{at } x=2\,; \qquad \text{and} \qquad \textbf{(iv)}\quad f(x) \;=\; \sqrt{x}\,, \quad \text{at } x=9\,.$$

b) Let $f(x) = x^2$.

(i) Compute the slope of the secant line joining the points on the graph of f whose x coordinates are $x = -2$ and $x = -1.9$.

(ii) Use calculus to compute the slope of the line that is tangent to the graph when $x = -2$, and compare with the slope found in part **(i)**.

c) Let $f(x) = x(x-1)^{-1}$.

(i) Compute the slope of the secant line joining the points on the graph of f whose x coordinates are $x = -1$ and $x = \frac{1}{2}$.

(ii) Use calculus to compute the slope of the tangent line to the graph of $f(x)$ at $x = -1$, and compare with the slope found in part **(i)**.

d) Let $f(x) = x(1-2x)$.

(i) Find the average rate of change of $f(x)$ with respect to x as x changes from $x = 0$ to $x = \frac{1}{2}$.

(ii) Use calculus to find the instantaneous rate of change of $f(x)$ at $x = 0$, and compare with the average rate found in part **(i)**.

Problem 4.2: Basic concepts of differentiation - II.

a) Differentiate the given functions $f(x)$ with respect to x:

$$\text{(i)}\quad f(x) \;=\; x^{3.7}\,, \qquad \text{(ii)}\quad f(x) \;=\; x^{7/3}\,, \qquad \text{(iii)}\quad f(x) \;=\; 3x^5 - 4x^3 + 9x - 6\,,$$

as well as

$$\text{(iv)}\quad f(x) \;=\; \sqrt{x^3} + \frac{1}{\sqrt{x^3}}\,, \qquad \text{and} \qquad \text{(v)}\quad f(x) \;=\; \frac{x^5 - 4x^2}{x^3} \quad [\textit{Hint}\text{: divide first}]\,.$$

b) Find the equation of the line that is tangent to the graph of the given function at the specified point.

(i) $f(x) = -x^3 - 5x^2 + 3x - 1$ at the point $(-1, -8)$.

(ii) $f(x) = (x^2 - x)(3 + 2x)$ at the point $(-1, 2)$.

(iii) $f(x) = 1 - \frac{1}{x} + \frac{2}{\sqrt{x}}$ at the point $(4, \frac{7}{4})$.

(iv) $f(x) = 2x^4 - \sqrt{x} + \frac{3}{x}$ at the point $(1, 4)$.

c) Sketch the graph of a function f that is continuous on its domain $(-5, 5)$ and where $f(0) = 1$, $f'(-2) = 0$, $\lim_{x\to -5^+} f(x) = \infty$, and $\lim_{x\to 5^-} f(x) = 3$.

d) Sketch the graph of a function f where the domain is $(-2, 2)$, $f'(0) = -2$, $\lim_{x\to 2^-} f(x) = \infty$, f is continuous at all numbers in its domain except ± 1, and f is odd.

Problem 4.3: Applications in business and economics.

a) A manufacturer determines that when x hundred units of a particular commodity are produced, the profit will be

$$P(x) \;=\; 4000 \cdot (15 - x)(x - 2)\,.$$

(i) Find $P'(x)$.

(ii) Determine where $P'(x) = 0$. What is the significance of the level of production x_m where this occurs?

b) Mariam, the business manager of a company that produces in-ground outdoor spas, determines that the cost of producing x spas is C thousand GEL, where

$$C(x) \;=\; 0.04x^2 + 2.1x + 60\,.$$

(i) If Mariam decides to increase the level of production from $x = 10$ to $x = 11$ spas, what is the corresponding average rate of change of cost?

(ii) Next, Mariam computes the instantaneous rate of change of cost with respect to the level of production x. What is this rate when $x = 10$, and how does the instantaneous rate compare with the average rate found in part **(i)**? By raising the level of production from $x = 10$, should Mariam expect the cost to increase or decrease?

c) At a certain factory, it is determined that an output of Q units is to be expected when L worker-hours of labor are employed, where

$$Q(L) \;=\; 3100\sqrt{L}\,.$$

(i) Find the average rate of change of output as the labor employment changes from $L =$ 3025 worker-hours to 3100 worker-hours.

(ii) Use calculus to find the instantaneous rate of change of output with respect to labor level when $L = 3025$.

d) The demand for a particular commodity is given by $D(x) = -35x + 200$; that is, x units will be sold (demanded) at a price of $p = D(x)$ GEL per unit.

(i) Consumer expenditure $E(x)$ is the total amount of money consumers pay to buy x units. Express consumer expenditure E as a function of x.

(ii) Find the average rate of change in consumer expenditure as x changes from $x = 4$ to $x = 5$.

(iii) Use calculus to find the instantaneous rate of change of expenditure with respect to x when $x = 4$. Is expenditure increasing or decreasing when $x = 4$?