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Week 5

Advanced Differentiation Rules

This exercise sheet consists of two parts: at first additional exercises are given the solutions of which are provided with the lecture slides and can serve you as further blueprints when solving similar tasks. Then, the actual homework assignments are stated. Please, hand-in your results of the homework assignments through MSTeams at the date and time specified in MSTeams.

Additional Exercises (see the lecture slides for solutions):

Exercise 5.1: Differentiate the following functions:

$$\textbf{a)}\quad f(x) = \tfrac{7}{4}x^2 - 3x + 12\,, \quad \textbf{b)}\quad f(x) = x^{-3/5} + x^4\,, \quad \textbf{c)}\quad f(x) = \sqrt{x} - 2\mathrm{e}^x\,,$$

Exercise 5.2: Figure 1 shows the graphs of four functions. One is the position function $s(t)$ of a car, one is the velocity $v(t) = \dot{s}(t)$ of the car, one is its acceleration $a(t) = \ddot{s}(t)$, and one is its jerk $j(t) = s^{(3)}(t)$. Identify each curve and explain your choices.

Exercise 5.3: Let $f(x) = \sqrt{x} \cdot g(x)$, where $g(4) = 2$ and $g'(4) = 3$. Determine the value of $f'(4)$.

Exercise 5.4: Differentiate the following functions:

$$\textbf{a)}\quad f(x) = (\mathrm{e}^x + 2)(2\mathrm{e}^x - 1)\,, \qquad \text{and} \qquad \textbf{b)}\quad f(x) = \frac{6x^4 - 5x}{x + 1}\,.$$

Exercise 5.5: If f is a differentiable function, find an expression for the derivative of $y = F(x) = \frac{f(x)}{x^2}$.

Homework Assignment:

Problem 5.1: The derivative of the exponential function and rates of changes.

a) Differentiate the given functions.

$$\textbf{(i)}\quad f(x) = x^{2.4} + \mathrm{e}^{2.4}\,, \qquad \textbf{(ii)}\quad f(x) = \mathrm{e}^x + x^{\mathrm{e}}\,,$$

as well as

$$\textbf{(iii)}\quad f(x) = \sqrt{x} - 4\mathrm{e}^x\,, \qquad \text{and} \qquad \textbf{(iv)}\quad f(x) = \mathrm{e}^{x+1} + 1\,.$$

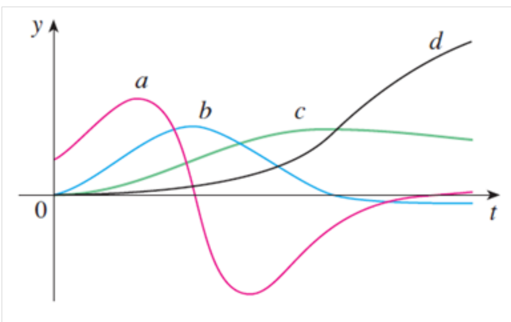

Figure 1: Sketches to be used in exercise 5.2.

b) Tangents to a graph.

(i) For what value of x does the graph of $f(x) = \mathrm{e}^x - 2x$ have a horizontal tangent?

(ii) Show that the curve $y = f(x) = -2\mathrm{e}^x + 3x + 5x^3$ has no tangent line with slope 2.

c) Find the rate of change of the given function $f(x)$ with respect to x for the prescribed value $x = c$.

$$\textbf{(i)} \quad f(x) = 2x^4 + 3x + 1, \quad \text{at } x = -1; \qquad \textbf{(ii)} \quad f(x) = x - \sqrt{x} + \frac{1}{x^2}, \quad \text{at } x = 1;$$

as well as

$$\textbf{(iii)} \quad f(x) = \frac{x + \sqrt{x}}{x}, \quad \text{at } x = 1; \qquad \text{and} \qquad \textbf{(iv)} \quad f(x) = \frac{2}{x} - x\sqrt{x}, \quad \text{at } x = 1.$$

d) Find the relative rate of change of $f(x)$ with respect to x for the prescribed value $x = c$.

$$\textbf{(i)} \quad f(x) = 2x^3 - 5x^2 + 4, \quad \text{at } x = 1; \qquad \textbf{(ii)} \quad f(x) = x + \frac{1}{x}, \quad \text{at } x = 1;$$

as well as

$$\textbf{(iii)} \quad f(x) = x\sqrt{x} + x^2, \quad \text{at } x = 4; \qquad \text{and} \qquad \textbf{(iv)} \quad f(x) = (4 - x)x^{-1}, \quad \text{at } x = 3.$$

Problem 5.2: Product and quotient rules as well as higher-order derivatives.

a) Differentiate the given functions.

$$\textbf{(i)} \quad f(x) = (2x + 1)(3x^2 - 2), \qquad \textbf{(ii)} \quad f(x) = \tfrac{1}{3}\left(x^5 - 2x^3 + 1\right)\left(x - \frac{1}{x}\right),$$

as well as

$$\textbf{(iii)} \quad f(x) = \frac{(x^2 + x + 1)(4 - x)}{2x - 1}, \qquad \text{and} \qquad \textbf{(iv)} \quad f(x) = \frac{x^2 + \sqrt{x}}{2x + 5}.$$

b) Find all points on the graph of the given function where the tangent line is horizontal.

$$\textbf{(i)} \quad f(x) = (x - 1)(x^2 - 8x + 7), \qquad \textbf{(ii)} \quad f(x) = (x + 1)(x^2 - x - 2),$$

as well as

$$\textbf{(iii)} \quad f(x) = \frac{x^2 + x - 1}{x^2 - x + 1}, \qquad \text{and} \qquad \textbf{(iv)} \quad f(x) = \frac{x + 1}{x^2 + x + 1}.$$

c) The normal line to the curve $y = f(x)$ at the point P with coordinates $(x_0, f(x_0))$ is the line perpendicular to the tangent line at P. Find an equation for the normal line to the given curve at the prescribed point.

(i) $f(x) = \frac{2}{x} - \sqrt{x}$ at the point $(0, -5)$.

(ii) $f(x) = (x+3)(1-\sqrt{x})$ at the point $(1, 0)$.

d) Find the second derivative of the given function. In each case, use the appropriate notation for the second derivative and simplify your answer. (Don't forget to simplify the first derivative as much as possible before computing the second derivative.)

$$\textbf{(i)} \quad f(x) = 5x^{10} - 6x^5 - 27x + 4, \qquad \textbf{(ii)} \quad f(x) = \frac{2}{3x} - \sqrt{2x} + \sqrt{2x} - \frac{1}{6\sqrt{x}},$$

as well as

$$\textbf{(iii)} \quad f(x) = \left(x^2 - x\right)\left(2x - \frac{1}{x}\right), \qquad \text{and} \qquad \textbf{(iv)} \quad f(x) = \left(x^3 + 2x - 1\right)(3x + 5).$$

Problem 5.3: Derivatives involving trigonometric functions.

a) Differentiate the given functions.

$$\textbf{(i)} \quad f(x) = x^2 \sin(x), \qquad \textbf{(ii)} \quad f(x) = \mathrm{e}^x \cos(x),$$

as well as

$$\textbf{(iii)} \quad f(x) = 5\cos(x) + x^2 \sin(x), \qquad \text{and} \qquad \textbf{(iv)} \quad f(x) = \sin(x)\cos(x).$$

b) Find an equation of the tangent line to the curve at the given point.

(i) $f(x) = \sin(x) + \cos(x)$ at $(0, 1)$.

(ii) $f(x) = x + \tan(x)$ at (π, π).

c) For what values of x does the graph of f have a horizontal tangent?

$$\textbf{(i)} \quad f(x) = x + 2\sin(x), \qquad \text{and} \qquad \textbf{(ii)} \quad f(x) = \mathrm{e}^x \cos(x).$$

d) Find the given derivative

$$\frac{\mathrm{d}^{99}}{\mathrm{d}x^{99}}(\sin(x))$$

by finding the first few derivatives and observing the pattern that occurs.

Problem 5.4: Applications in business and economics.

a) The manager of the Many Facets jewelry store models total sales by the function

$$S(t) = \frac{2000\,t}{4 + 0.3\,t},$$

where t is the time (years) since the year 2018 and S is measured in thousands of GEL.

(i) At what rate were sales changing in the year 2020?

(ii) What happens to sales in the long run (that is, as $t \to \infty$)?

b) A company manufactures a 'thin' BlueRay burner kit that can be plugged into personal computers. The marketing manager determines that t weeks after an advertising campaign begins, $P(t)$ percent of the potential market is aware of the burners, where

$$P(t) = 100\left(\frac{t^2 + 5t + 5}{t^2 + 10t + 30}\right).$$

(i) At what rate is the market percentage $P(t)$ changing with respect to time after 5 weeks? Is the percentage increasing or decreasing at this time?

(ii) What happens to the percentage $P(t)$ in the long run, that is, as $t \to \infty$? What happens to the rate of change of $P(t)$ as $t \to \infty$?

c) An efficiency study of the morning shift at a certain factory indicates that an average worker arriving on the job at 8:00 A.M. will have produced $Q(t) = -t^3 + 8t^2 + 15t$ units t hours later.

(i) Compute the worker's rate of production $R(t) = Q'(t)$.

(ii) At what rate is the worker's rate of production changing with respect to time at 9:00 A.M.?

d) The manager of a company estimates that it will cost 10000 GEL to produce 400 units of her product 1 year from now and that all those units can then be sold at a price of 30 GEL per unit. She also estimates that in 1 year, the price will be increasing at the rate of 75 cents per unit per month, while the level of production will be decreasing at the rate of 2 units per month and the cost will stay constant.

(i) If x is the level of production at time t, where $t = 0$ is 1 year from now, what is the profit $P(x)$? Find the rate at which the profit will be changing 1 year from now with respect to x. Will the profit be increasing or decreasing at that time?

(ii) At what rate will the average profit $\frac{P(x)}{x}$ be changing 1 year from now? Will the average profit be increasing or decreasing at that time?