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Fall Term
Week 6

Applications of Differentiation I

This exercise sheet consists of two parts: at first additional exercises are given the solutions of which are provided with the lecture slides and can serve you as further blueprints when solving similar tasks. Then, the actual homework assignments are stated. Please, hand-in your results of the homework assignments through MSTeams at the date and time specified in MSTeams.

Additional Exercises (see the lecture slides for solutions):

Exercise 6.1: Apply the chain rule to differentiate the following functions:

a) $f(x) = (x^3 - 1)^{100}$, **b)** $f(x) = (2x+1)^5(x^3 - x + 1)^4$, **c)** $f(x) = 3x + x^2\cos(x)$,

as well as

d) $f(x) = \mathrm{e}^x(\tan(x) - x)$ and **e)** $f(x) = \dfrac{\sin(x)}{1+\tan(x)}$.

Exercise 6.2: Implicit differentiation.

a) Let $\sin(x+y) = y^2\cos(x)$. Find y'.

b) Let $x^4 + y^4 = 16$. Find y''.

Exercise 6.3: L'Hospitals rules.

a) Calculate $\lim_{x\to\infty} \frac{\mathrm{e}^x}{x^2}$.

b) Calculate $\lim_{x\to 0+} (1+\sin(4x))^{\cot(x)}$.

Exercise 6.4: Compare the values of Δy and $\mathrm{d}y$ if $y = f(x) = x^3 + x^2 - 2x + 1$ and x changes **a)** from 2 to 2.05 and **a)** from 2 to 2.01.

Homework Assignment:

Problem 6.1: Application of the Chain Rule.

a) Differentiate the given functions and simplify your answer:

(i) $f(x) = (2\mathrm{e}^{5x} + 3)^{1.4}$, **(ii)** $f(x) = \dfrac{1}{\sqrt{4x^2+1}}$, **(iii)** $f(x) = (x+2)^3(2\sin(x) - 1)$,

as well as

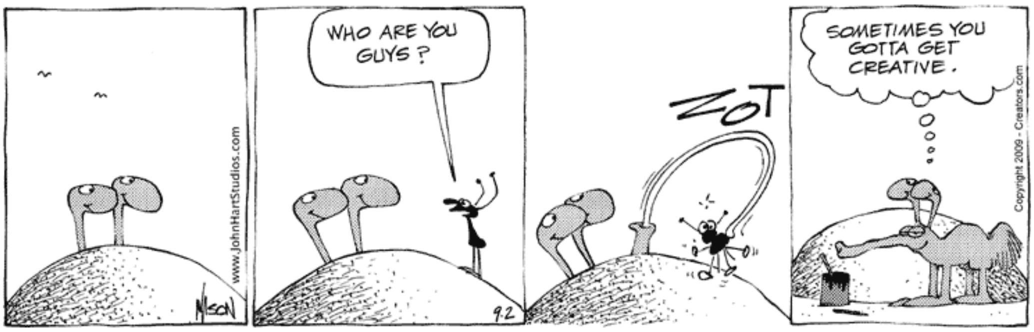
(iv) $f(x) = \sqrt{1 + \dfrac{1}{3x}}$, and **(v)** $f(x) = \dfrac{1-5x^2}{\sqrt[3]{3+2x}}$.

b) **(i)** Find $h'(-1)$ if $g(-1) = -1$ and $g'(-1) = 1$, where

$$h(x) = \left(3g^2(x) + 4g(x) + 2\right)^5 (g(x) + x) .$$

(ii) Let $g(0) = 3$ and $g'(0) = -2$. Find $h'(0)$ if

$$h(x) = \left(\frac{g(x) - x}{3 + g(x)}\right)^2 .$$



c) Find all values of so that the tangent line to the graph of $f(x)$ at $(c, f(c))$ will be horizontal:

$$\textbf{(i)}\quad f(x) \;=\; x^3(2x^2+x-3)^2\,, \qquad \text{and} \qquad \textbf{(ii)}\quad f(x) \;=\; \sqrt{x^2-4x+5}\,.$$

d) The value V (in thousands of GEL) of an industrial machine is modeled by

$$V(N) \;=\; \left(\frac{3N+430}{N+1}\right)^{2/3}\,,$$

where N is the number of hours the machine is used each day. Suppose further that usage varies with time in such a way that where t is the number of months the machine has been in operation.

(i) How many hours per day will the machine be used 9 months from now? What will be the value of the machine at this time?

(ii) At what rate is the value of the machine changing with respect to time 9 months from now? Will the value be increasing or decreasing at this time?

Problem 6.2: Implicit differentiation.

a) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ by implicit differentiation:

$$\textbf{(i)}\quad x^2+y^2 \;=\; 25\,, \qquad \textbf{(ii)}\quad y^2+2xy^2-3x+1 \;=\; 0\,,$$

as well as

$$\textbf{(iii)}\quad f(x) \;=\; (2x+y)^3 \;=\; x\,, \qquad \text{and} \qquad \textbf{(iv)}\quad f(x) \;=\; (3xy^2+1)^4 \;=\; 2x-3y\,.$$

b) Find the equation of the tangent line to the given curve at the specified point:

$$\textbf{(i)}\quad x^2 \;=\; y^3\,, \quad \text{at } (8,4)\,; \qquad \textbf{(ii)}\quad x^2y^3-2xy \;=\; 6x+y+1\,, \quad \text{at } (0,-1)\,;$$

as well as

$$\textbf{(iii)}\quad (1-x+y)^3 \;=\; x+7\,, \quad \text{at } (1,2)\,; \qquad \text{and} \qquad \textbf{(iv)}\quad xy \;=\; 2\,, \quad \text{at } (2,1)\,.$$

c) Find all points (both coordinates) on the given curve where the tangent line is either horizontal or vertical:

$$\textbf{(i)}\quad x+y^2 \;=\; 9\,, \quad \textbf{(ii)}\quad x^2+xy+y^2 \;=\; 3\,, \quad \text{and} \quad \textbf{(iii)}\quad \frac{y}{x}-\frac{x}{y} \;=\; 5\,.$$

d) At a certain factory, output is given by $Q = 60K^{1/3}L^{2/3}$ units, where K is the capital investment (in thousands of GEL) and L is the size of the labor force, measured in worker-hours. If output is kept constant, at what rate is capital investment changing at a time when $K = 8$, $L = 1000$, and L is increasing at the rate of 25 worker-hours per week?

Remark: Output functions of the general form $Q = AK^{\alpha}L^{1-\alpha}$, where A and α are constants with $0 \le \alpha \le 1$, are called **Cobb-Douglas production functions**. Such functions appear in examples and exercises throughout our courses Calculus I and Calculus II.

Problem 5.3: Linear Approximations & L'Hospital's Rules.

a) Find the linearization $L(x)$ of the given functions at the mentioned numbers a:

$$\textbf{(i)}\quad f(x) = x^3 - x^2 + 3, \quad \text{at } a = 2; \qquad \textbf{(ii)}\quad f(x) = \sin(x), \quad \text{at } a = \tfrac{1}{6}\pi;$$

as well as

$$\textbf{(iii)}\quad f(x) = \sqrt{x}, \quad \text{at } a = 4; \qquad \text{and} \qquad \textbf{(iv)}\quad f(x) = 2^x, \quad \text{at } a = 0.$$

b) Suppose that we don't have a formula for $f(x)$ but we know that $f(2) = -4$ and $f'(x) = \sqrt{x^2+5}$ for all x.

(i) Use a linear approximation to estimate $f(1.095)$ and $f(2.05)$.

(ii) Are your estimates in part **(i)** too large or too small? Explain.

c) Find the limit. Use L'Hospital's Rules where appropriate. If there is a more elementary method, consider using it. If L'Hospital's Rule doesn't apply, explain why.

$$\textbf{(i)}\quad \lim_{x\to 4} \frac{x^2-2x-8}{x-4}, \qquad \textbf{(ii)}\quad \lim_{x\to\infty} \frac{\ln(\sqrt{x})}{x^2}, \qquad \textbf{(iii)}\quad \lim_{x\to 0} \frac{x\cdot 3^x}{3^x-1},$$

as well as

$$\textbf{(iv)}\quad \lim_{x\to 0+} \frac{\ln(x)}{x}, \qquad \text{and} \qquad \textbf{(v)}\quad \lim_{x\to\pi} \frac{1+\cos(x)}{1-\cos(x)}.$$

d) Find the limit. Use L'Hospital's Rules where appropriate. If there is a more elementary method, consider using it. If L'Hospital's Rule doesn't apply, explain why.

$$\textbf{(i)}\quad \lim_{x\to 0} \frac{\mathrm{e}^x - \mathrm{e}^{-x} - 2x}{x - \sin(x)}, \qquad \textbf{(ii)}\quad \lim_{x\to\infty} \sqrt{x}\,\mathrm{e}^{-x/2}, \qquad \textbf{(iii)}\quad \lim_{x\to\infty} (x - \ln(x)),$$

as well as

$$\textbf{(iv)}\quad \lim_{x\to 0+} (1+\sin(3x))^{1/x}, \qquad \text{and} \qquad \textbf{(v)}\quad \lim_{x\to\infty} \left(\frac{2x-3}{2x+5}\right)^{2x+1}.$$

Problem 5.4: Applications in business and economics

a) After t weeks, a factory is producing $N(t)$ thousand BlueRay players, where

$$N(t) = \frac{2t}{t^2+3t+12}.$$

At what rate is the production level changing after 4 weeks? Is production increasing or decreasing at this time?

b) At a certain factory, output Q is related to inputs x and y by the equation

$$Q = 2x^3 + 3x^2y^2 + (1+y)^3.$$

If the current levels of input are $x = 30$ and $y = 20$, use calculus to estimate the change in input y that should be made to offset a decrease of 0.8 unit in input x so that output will be maintained at its current level.

c) If an initial amount A_0 of money is invested at an interest rate r compounded n times a year, the value of the investment after t years is

$$A = A_0\left(1+\frac{1}{n}\right)^{nt}.$$

If we let $n\to\infty$, we refer to the **continuous compounding** of interest. Use L'Hospital's Rule to show that if interest is compounded continuously, then the amount after t years is

$$A = A_0\mathrm{e}^{rt}.$$

d) Populations of consumers of very useful devices (like smart phones) initially grow exponentially but eventually level off as the total population or the relevant population share is getting exhausted (e.g. that with the relevant buying power). Equations of the form

$$P(t) = \frac{M}{1 + Ae^{-kt}},$$

where M, A, and k are positive constants, are called **logistic equations** and are often used to model such populations. Here M is called the carrying capacity and represents the maximum population size that can be supported, and $A = \frac{M-P_0}{P_0}$, where P_0 is the initial population.

(i) Compute $\lim_{t\to\infty} P(t)$. Explain why your answer is to be expected.

(ii) Compute $\lim_{M\to\infty} P(t)$. (Note, that A is defined in terms of M.) What kind of function is your result?