Kutaisi International University	Calculus I for Management

Prof. Dr. Giorgi Chelidze
Prof. Dr. Malkhaz Shashiashvili
Prof. Dr. Dr. h.c. Florian Rupp

Fall Term
Week 14

Applications of Integration in Economics

This exercise sheet consists of two parts: at first additional exercises are given the solutions of which are provided with the lecture slides and can serve you as further blueprints when solving similar tasks. Then, the actual homework assignments are stated. Please, hand-in your results of the homework assignments through MSTeams at the date and time specified in MSTeams.

Additional Exercises (see the lecture slides for solutions):

Exercise 14.1: Find the Gini index for the given Lorenz curve:

$$L_1(x) = x^3 \qquad \text{and} \qquad L_2(x) = \tfrac{2}{3}x^{3.7} + \tfrac{1}{3}x\,.$$

Exercise 14.2: Find the average value of f on $[0, 8]$, where the graph of f is shown in figure 1.

Exercise 14.3: Graph the function $f(x) = x^2 - 1$ on $[0, \sqrt{3}]$ and find its average value over the given interval.

Exercise 14.4: An annuity pays a continuous income stream of M GEL per year into an account that pays interest at an annual rate r compounded continuously for a term of T years.

a) Show that the future value FV of the annuity is

$$FV = \tfrac{M}{r}\left(\mathrm{e}^{rT} - 1\right)\,.$$

b) Show that the present value PV of the annuity is

$$PV = \tfrac{M}{r}\left(1 - \mathrm{e}^{-rT}\right)\,.$$

Exercise 14.5: For the consumers' demand functions $D(q) = 2(64 - q^2)$ GEL per unit find the total amount of money consumers are willing to spend to obtain $q_0 = 6$ units of the commodity.

Exercise 14.6: Let $p = D(q) = 2(64 - q^2)$ be the price (GEL per unit) at which q units of a particular commodity will be demanded by the market (that is, all q units will be sold at this price). Find the price $p_0 = D(q_0)$ at which $q_0 = 6$ units will be demanded and compute the corresponding consumers' surplus CS.

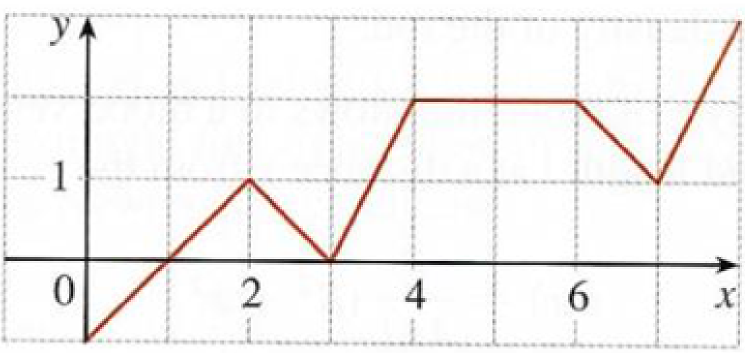

Figure 1: Graph of the function used in exercise 14.2.

Homework Assignment:

Problem 14.1: The average value of a function, net excess profit & Lorenz curves

a) Find the average value of the function on the given interval.

(i) $f(x) = 3x^2 + 8x$ on $[-1, 2]$.

(ii) $f(x) = 3\cos(x)$ on $[-\frac{1}{2}\pi, \frac{1}{2}\pi]$.

(iii) $f(x) = \dfrac{x}{\sqrt{3 + x^2}}$ on $[1, 3]$.

(iv) $f(x) = \mathrm{e}^{\sin(x)}\cos(x)$ on $[0, \frac{1}{2}\pi]$.

b) A manufacturer supplies $S(p) = 0.5p^2 + 3p + 7$ hundred units of a certain commodity to the market when the price is p GEL per unit. Find the average supply as the price varies from $p = 2$ GEL to $p = 5$ GEL.

c) Suppose that t years from now, one investment plan will be generating profit at the rate of $P_1(t) = 130 + t^2$ thousand GEL per year, while a second investment will be generating profit at the rate of $P_2(t) = 306 + 5t$ thousand GEL per year.

(i) For how many years does the rate of profitability of the second investment exceed that of the first?

(ii) Compute the net excess profit assuming that you invest in the second plan for the time period determined in part **(i)**.

(iii) Sketch the rate of profitability curves $y = P_1(t)$ and $y = P_2(t)$, and shade the region whose area represents the net excess profit computed in part **(ii)**.

d) In a certain country, it is found that the distribution of income for lawyers is given by the Lorenz curve $y = L_1(x)$, where

$$L_1(x) = \tfrac{4}{5}x^2 + \tfrac{1}{5}x$$

while that of surgeons is given by $y = L_2(x)$, where

$$L_2(x) = \tfrac{5}{8}x^4 + \tfrac{3}{8}x$$

Compute the Gini index for each Lorenz curve. Which profession has the more equitable income distribution?

Problem 14.2: Future value & present value of an income flow

a) Luisa, a 2 USD million state lottery winner, is given a 250000 check now and a continuous income flow at the rate of 200000 per year for 10 years. If the prevailing rate of interest is 5% per year compounded continuously, is this a good deal for Luisa or not? Explain.

b) The management of a national chain of fast-food outlets is selling a 10-year franchise in Samtredia. Past experience in similar localities suggests that t years from now the franchise will be generating profit at the rate of $f(t) = 250000$ GEL per year. If the prevailing annual interest rate remains fixed at 4% compounded continuously, what is the present value of the franchise?

c) Oil is being pumped from an oil field t years after its opening at the rate of $P'(t) = 1.3e^{0.04t}$ billion barrels per year. The field has a reserve of 20 billion barrels, and the price of oil holds steady at 112 USD per barrel.

(i) Find $P(t)$, the amount of oil pumped from the field at time t. How much oil is pumped from the field during the first 3 years of operation? The next 3 years?

(ii) For how many years T does the field operate before it runs dry?

(iii) If the prevailing annual interest rate stays fixed at 5% compounded continuously, what is the present value of the continuous income stream $V = 112P'(t)$ over the period of operation of the field $0 \leq t \leq T$?

(iv) If the owner of the oil field decides to sell on the first day of operation, do you think the present value determined in part **(iii)** would be a fair asking price? Explain your reasoning.

Problem 14.3: Consumer willingness to spend & consumer's surplus

a) For the following consumers' demand functions $D(q)$ find the total amount of money consumers are willing to spend to obtain q_0 units of the commodity and then sketch the demand curve and interpret the consumer willingness to spend as an area.

(i) $D(q) = \frac{300}{(0.1q+1)^2}$ GEL per unit; $q_0 = 5$ units.

(ii) $D(q) = 50e^{-0.04q}$ GEL per unit; $q_0 = 15$ units.

b) Let the demand and supply functions, $D(q)$ and $S(q)$, for a particular commodity be given as in **(i)** and **(ii)**. Specifically, q units of the commodity will be demanded (sold) at a price of $p = D(q)$ GEL per unit, while q units will be supplied by producers when the price is $p = S(q)$ GEL per unit. In each case, find the equilibrium price p_e (where supply equals demand) and then find the consumers' surplus and the producers' surplus at equilibrium.

(i) $D(1) = 131 - \frac{1}{3}q^2$; $S(q) = 50 + \frac{2}{3}q^2$.

(ii) $D(q) = \sqrt{245 - 2q}$; $S(q) = 5 + q$.

c) A manufacturer of machinery parts determines that q units of a particular piece will be sold when the price is $p = 124 - 2q$ GEL per unit. The total cost of producing those q units is $C(q)$ GEL, where

$$C(q) = 2q^3 - 59q^2 + 4q + 7600\,.$$

(i) How much profit is derived from the sale of the q units at p GEL per unit? (Hint: First find the revenue $R = pq$; then find profit = revenue − cost.)

(ii) For what value of q is profit maximized?

(iii) Find the consumers' surplus for the level of production q_0 that corresponds to maximum profit.